arXiv:math/0409222v4 [math.OA] 1 Nov 2004

On C^*-algebras generated by some deformations of CAR relations

Daniil Proskurin, Yuriĭ Savchuk, and Lyudmila Turowska

ABSTRACT. We study the representations and enveloping C^*-algebras for Wick analogues of CAR and twisted CAR algebras. Realizations of the considered C^*-algebras are given as algebras of continuous matrix-functions satisfying certain boundary conditions.

Introduction

In this paper we study $*$-representations and enveloping C^*-algebras for some versions of the canonical anti-commutation relations.

Recall that the CAR algebra with d degrees of freedom is generated by $a_i,\ a_i^*$, $i = 1, \dots d$, and the relations

$$\begin{aligned} &a_i^* a_i + a_i a_i^* = 1,\ a_i^2 = 0,\ i = 1, \dots d, \\ &a_i^* a_j = -a_j a_i^*,\ a_j a_i = -a_i a_j,\ i \neq j. \end{aligned} \tag{0.1}$$

It is known that the Fock representation is the unique irreducible representation of (0.1) and the C^*-algebra generated by (0.1) is isomorphic to $M_{2^d}(\mathbb{C})$.

We consider an interpolation between CCR and CAR known as q-CCR, proposed by A.J.Macfarlane and L.C.Biedenharn for $d = 1$, see [**B, M3**] and by O.Greenberg, D.Fivel, M.Bozeiko and R.Speicher for general d, see [**BS, F, G**]. Namely, the higher-dimensional q-CCR have the following form

$$a_i^* a_j = \delta_{ij} 1 + q a_j a_i^*, i = 1, \dots, d,\ q \in (-1, 1). \tag{0.2}$$

Another well-known deformation of CAR, called twisted CAR, was introduced and studied by W.Pusz, see [**P2**]. The twisted CAR $*$-algebra (TCAR) is generated

1991 *Mathematics Subject Classification.* Primary 46L55, 47C05. Secondary 81S05.

Key words and phrases. Canonical anti-commutation relations, enveloping C^*-algebra, algebra of matrix-functions.

The work was completed when the first author was visiting Chalmers University of Technology in Göteborg, Sweden. The visit was supported by a grant from the Swedish Royal Academy of Sciences as a part of program of cooperation with former Soviet Union and by STINT..

Lyudmila Turowska was partially supported by Swedish Research Council.

by $a_i, a_i^*,\ i = 1, \ldots, d$, subject to the following relations

$$
\begin{gathered}
a_i^* a_i = 1 - a_i a_i^* - (1-\mu^2) \sum_{j<i} a_j a_j^*,\ i = 1, \ldots, d,\ \mu \in (0,1), \\
a_i^* a_j = -\mu a_j a_i^*,\ a_j a_i = -\mu a_i a_j,\ i < j, \\
a_i^2 = 0,\ i = 1, \ldots, d.
\end{gathered} \tag{0.3}
$$

The Fock representation is the unique irreducible representation of TCAR, and as in the non-deformed case the C^*-algebra generated by TCAR coincides with $M_{2^d}(\mathbb{C})$.

The Wick analogue of TCAR (denoted below as WTCAR) one obtains from TCAR taking away the relations between $a_i,\ a_j$. This algebra was studied in [**JSW, P1**]. In particular, it was shown that in any representation of WTCAR the relations

$$a_j a_i = -\mu a_i a_j,\ i < j,\ a_i^2 = 0,\ i = 1, \ldots, d-1,$$

are satisfied and the irreducible representations of WTCAR were classified.

The q-CCR and the WTCAR with $d = 1$ are closely related with a $*$-algebra known as the quantum disk. That is a $*$-algebra generated by a and a^* satisfying the relation

$$a^* a - q a a^* = (1-q),\ q \in (-1, 1). \tag{0.4}$$

The family of C^*-algebras D_q generated by (0.4) was studied by many authors, see for example, [**NN**].

If in (0.2) we put $q = -1$ and in (0.3) put $\mu = 1$ we get the Wick analogue of CAR, i.e. the $*$-algebra generated by relations of the form

$$
\begin{gathered}
a_i^* a_i + a_i a_i^* = 1,\ i = 1, \ldots d, \\
a_i^* a_j = -a_j a_i^*,\ i \neq j.
\end{gathered} \tag{0.5}
$$

In [**JW**] P.E.T. Jørgensen and R.F. Werner studied representations of WCAR using the representation theory of Clifford algebras. In particular, it was shown that in the irreducible representations of (0.5), for any pair (i, j), one has

$$a_i a_j + a_j a_i = y_{ij},\ \text{where } y_{ij} \in \mathbb{C},\ \|(y_{ij})\| \leq 1.$$

It was also stated in [**JW**] that for any $Y = (y_{ij}),\ y_{ji} = y_{ij} \in \mathbb{C}$ with $\|Y\| \leq 1$ there exists an irreducible representation of (0.5) with

$$a_i a_j + a_j a_i = y_{ij}. \tag{0.6}$$

Moreover, the C^*-algebra $\widehat{\mathcal{E}}(-1, Y)$ generated by relations (0.5), (0.6) was shown to be either isomorphic to $M_n(\mathbb{C})$ or to $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ for appropriate $n \in \mathbb{N}$. In particular, this fact implies that for any fixed $Y,\ \|Y\| \leq 1,\ Y^T = Y$, there are at most two non-equivalent irreducible representations of relations (0.5), (0.6). Here we continue the study of C^*-algebras associated with WCAR.

Our paper is organized as follows. In Section 1 we give some definitions and facts used in the paper and fix notation.

In Section 2 we obtain a realization of WTCAR algebra as algebra of continuous matrix-functions satisfying some boundary conditions. Analysis of the case $d = 1$ is crucial. Note that for $d = 1$ we get, up to normalization, the quantum disk with $q = -1$, called also the "non-commutative circle". The C^*-algebra D_{-1} was studied in [**NN**]. In particular it was shown that D_{-1} can be faithfully embedded

into the algebra, $C(D^2 \to M_2(\mathbb{C}))$, of continuous matrix-functions on the unit disk. We make this result more precise and show that D_{-1} is isomorphic to an algebra of continuous 2×2 matrix-functions on the disk D^2 satisfying certain boundary conditions on $S^1 = \partial D^2$. Note that it is more convenient for us to use the embedding of D_{-1} into $M_2(C(D^2))$ in the form different from the one presented in [**NN**].

In the Section 3 we study representations of WTCAR with $d = 2$ using a dynamical systems technique, see [**OS**]. For any $y \in \mathbb{C}$, $|y| \leq 1$ we give a parameterization of the unitary equivalence classes of irreducible representations and describe the C^*-algebra $\mathcal{E}(-1, y)$ generated by relations

$$\begin{aligned} a_i^* a_i + a_i a_i^* &= 1, \ i = 1, 2, \\ a_1^* a_2 &= -a_2 a_1^*, \\ a_2 a_1 + a_1 a_2 &= y. \end{aligned} \tag{0.7}$$

Further we prove that the set of the isomorphism classes of $\mathcal{E}(-1, y)$ consists of three elements: $[\mathcal{E}(-1, 0)]$, $[\mathcal{E}(-1, y), \ 0 < |y| < 1]$ and $[\mathcal{E}(-1, y), \ |y| = 1]$, where by $[\cdot]$ we denote the class of isomorphic algebras. We also describe the C^*-algebras $\mathcal{E}(\varepsilon)$, $0 < \varepsilon < 1$, defined by (0.7) where y takes any value from the set $\{\varepsilon \leq |y| \leq 1\}$. The isomorphism question is also discussed.

In Section 4 we describe the enveloping C^*-algebra of WCAR.

1. Preliminaries

In this section, for convenience of the reader, we fix some notation and recall necessary definitions and facts used in the paper.

Let $\mathcal{A}$ be a $*$-algebra, having at least one representation. Then a pair (A, ρ) of a C^*-algebra A and a homomorphism $\rho : \mathcal{A} \to A$ is called an *enveloping pair* for $\mathcal{A}$ if every irreducible representation $\pi : \mathcal{A} \to B(H)$ factors uniquely through A, i.e. there is a unique irreducible representation π_1 of the algebra A satisfying $\pi_1 \circ \rho = \pi$. The C^*-algebra A is called an *enveloping* for $\mathcal{A}$. An enveloping C^*-algebra for a $*$-algebra $\mathcal{A}$ is unique and exists iff the set of bounded representations of $\mathcal{A}$ is not empty and $\mathcal{A}$ is $*$-bounded, i.e. for any $a \in \mathcal{A}$ one can find $C_a > 0$ such that for any bounded representation, π, $\|\pi(a)\| \leq C_a$.

The following statement is a simple corollary of the non-commutative analogue of the Stone-Weierstrass theorem (see [**F, V**]).

THEOREM 1.1. *Let Y be a compact Hausdorff space. Let $C \subseteq B$ be subalgebras of $A = C(Y \to M_n(\mathbb{C}))$. For every pair $x_1, x_2 \in Y$ define $B(x_1, x_2)$ ($C(x_1, x_2)$ respectively) as :*

$$B(x_1, x_2) := \{(f(x_1), f(x_2)) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \ | f \ \in B\},$$

Then

$$B = C \Leftrightarrow B(y_1, y_2) = C(y_1, y_2), \forall y_1, y_2 \in Y.$$

For representations π_1, π_2 of $*$-algebra $\mathcal{A}$ on Hilbert spaces $\mathcal{H}(\pi_1)$ and $\mathcal{H}(\pi_2)$ respectively, let $C(\pi_1, \pi_2)$ be the space of intertwining operators

$$C(\pi_1, \pi_2) = \{c \in B(\mathcal{H}(\pi_2), \mathcal{H}(\pi_1)) : \pi_1(a)c = c\pi_2(a), \ a \in \mathcal{A}\}.$$

Note that $C(\pi_1, \pi_2) = \{0\}$ iff π_1, π_2 are disjoint, i.e. π_1, π_2 do not have unitary equivalent subrepresentations.

For a $*$-algebra $\mathcal{A} \subset B(\mathcal{H})$ we denote by $\mathcal{A}'$ its commutant, i.e.

$$\mathcal{A}' = \{c \in B(\mathcal{H}) : \ ca = ac, \ a \in \mathcal{A}\}.$$

In what follows we will identify $\mathbb{C}^n \otimes \mathbb{C}^m$ with $\mathbb{C}^{nm}$ in such way that for $A = (a_{ij}) \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ the matrix $A \otimes B$ is equal to $(a_{ij}B) \in M_{nm}(\mathbb{C})$.

2. Enveloping C^*-algebra for WTCAR

In this Section we give a realization of enveloping C^*-algebra for the $*$-algebra, $\mathcal{B}_\mu^{(d)}$, generated by WTCAR as algebra of continuous matrix-functions.

2.1. We first study the case $d = 1$. Evidently

$$\mathcal{B}_\mu^{(1)} = \mathbb{C}\langle a, a^* \mid a^*a + aa^* = 1\rangle. \tag{2.1}$$

Above we noted that this C^*-algebra is isomorphic to the non-commutative circle D_{-1} studied, in particular in [**NN**]. To get a realization of D_{-1} as continuous matrix-functions we use a classification of its irreducible representations up to unitary equivalence. We use method of dynamical systems presented in [**OS**] in order to obtain this classification. Let π be a representation of (2.1). We consider the polar decomposition of $\pi(a) = uc$. If π is irreducible then (2.1) implies that $\sigma(c^2) = \{x, 1-x\}$, $0 \le x \le \frac{1}{2}$, and $u^2 = e^{i\phi}\mathbf{1}$ if $x \ne 0$ and $u^2 = 0$, if $x = 0$. Moreover the eigenvalues of c^2 should have the same multiplicities (see [**OS**]) implying that the irreducible representations with $\sigma(c^2) \ne \{\frac{1}{2}\}$ are two-dimensional and the irreducible representations with $\sigma(c^2) = \{\frac{1}{2}\}$ are one-dimensional. Finally we have the following list of irreducible representations:

- 2-dimensional:

$$\pi_{x,\phi}(a) = \begin{pmatrix} 0 & e^{i\phi}\sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}, x \in [0, 1/2), \phi \in [0, 2\pi), \tag{2.2}$$

- 1-dimensional:

$$\rho_\phi(a) = \frac{e^{i\phi}}{\sqrt{2}}, \ \phi \in [0, 2\pi). \tag{2.3}$$

An alternative description can be found in [**M1, NN**].

Remark 2.1. Let

$$V(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\phi}{2}} & -e^{i\frac{\phi}{2}} \\ 1 & 1 \end{pmatrix}. \tag{2.4}$$

Then

$$V^*(\phi)\pi_{\frac{1}{2},\phi}(a)V(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & -e^{\frac{i\phi}{2}} \end{pmatrix}.$$

showing that any 1-dimensional representation can be obtained decomposing the representation $\pi_{\frac{1}{2},\phi}$ with some fixed ϕ into irreducible ones.

A result similar to one given in the next theorem can be found in [**M1**]. Here we anyway give a detailed proof of the statement, since it presents in the most transparent way an idea of the more tedious proofs of Theorems 3.2, 3.3, 3.5.

Theorem 2.2. *The C^*-algebra D_{-1} is isomorphic to the C^*-algebra*

$$A_1 = \{f \in C(D^2 \to M_2(\mathbb{C})) \mid V^*(\phi)f(e^{i\phi})V(\phi) \in \mathbb{C} \oplus \mathbb{C}, \forall \phi \in [0, 2\pi]\},$$

where $D^2 = \{z \in \mathbb{C} : \ |z| \le 1\}$, and $V(\phi)$ is given by the (2.4).

PROOF. Let $I_{\frac{1}{2}} = [0, 1/2]$ and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Firstly we give a "primary" realization of D_{-1}. We show, that D_{-1} is isomorphic to

$$A_0 = \{f \in C(I_{\frac{1}{2}} \times S^1 \to M_2(\mathbb{C})) \mid f(0, e^{i\phi}) = f(0, 1),\ \forall \phi \in [0, 2\pi],$$
$$V^*(\phi) f(1/2, e^{i\phi}) V(\phi) \text{ is diagonal}, \forall \phi \in [0, 2\pi]\},$$

Let $\widetilde{a} = \widetilde{a}(x, e^{i\phi}) : I_{\frac{1}{2}} \times S^1 \to M_2(\mathbb{C})$ be the function given by (2.2), i.e.

$$\widetilde{a}(x, e^{i\phi}) = \pi_{x,\phi}(a) = \begin{pmatrix} 0 & e^{i\phi}\sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}.$$

One can check that $\widetilde{a} \in A_0$. Let $\widehat{A_0}$ be the C^*-subalgebra of A_0 generated by $\widetilde{a}$. The isomorphism $\widehat{A_0} \simeq D_{-1}$ follows directly from the definition of enveloping pair.

To prove the equality $\widehat{A_0} = A_0$ we check the conditions of Theorem 1.1, i.e.

$$\widehat{A}_0((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})) = A_0((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})), \tag{2.5}$$
$$\forall (x_1, e^{i\phi_1}), (x_2, e^{i\phi_2}) \in I_{\frac{1}{2}} \times S^1.$$

Since $\widehat{A}_0((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})) \subset A_0((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2}))$ and these algebras are finite-dimensional, to prove (2.5) it is sufficient to show that their commutants are equal.

On the set $I_{\frac{1}{2}} \times S^1$ we introduce the equivalence

$$(x_1, e^{i\phi_1}) \sim (x_2, e^{i\phi_2}), \text{ iff } x_1 = x_2 = 0. \tag{2.6}$$

Note, that if $(x_1, e^{i\phi_1}) \nsim (x_2, e^{i\phi_2})$, then

$$C(\pi_{x_1,\phi_1}, \pi_{x_2,\phi_2}) = \{0\}.$$

Since

$$\widehat{A}_0((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})) = \left\{ \begin{pmatrix} \pi_{x_1,\phi_1}(b) & 0 \\ 0 & \pi_{x_2,\phi_2}(b) \end{pmatrix},\ b \in D_{-1} \right\},$$

then

$$\widehat{A}_0'((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})) = \left\{ \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},\ \Lambda_i \in \{\pi_{x_i,\phi_i}(b),\ b \in D_{-1}\}' \right\}.$$

The inclusion $\widehat{A}_0 \subset A_0$ implies that

$$A_0'((x_1, e^{i\phi_1}), (x_2, e^{i\phi_2})) = \left\{ \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},\ \Lambda_i \in A_0'(x_i, e^{i\phi_i}) \right\},$$

where

$$A_0(x, e^{i\phi}) = \{f(x, e^{i\phi}) \mid\ f \in A_0\}.$$

If $(x_1, e^{i\phi_1}) \sim (x_2, e^{i\phi_2})$ then $x_1 = x_2 = 0$ and $\pi_{x_1,\phi_1} = \pi_{x_2,\phi_2} = \pi_{0,1}$. In this case

$$\widehat{A}_0'((0,1),(0,1)) = \left\{(\Lambda_{ij}),\ \Lambda_{ij} \in \{\pi_{0,1}(b) \mid b \in D_{-1}\}',\ i,j = 1,2\right\}$$

and

$$A_0'((0,1),(0,1)) = \{(\Lambda_{ij}),\ \Lambda_{ij} \in A_0'(0,1),\ i,j = 1,2\}.$$

It is left to show that for any $(x, \phi) \in I_{\frac{1}{2}} \times [0, 2\pi)$

$$\{\pi_{x,\phi}(b),\ b \in D_{-1}\}' = A_0'(x, e^{i\phi}).$$

We consider two cases: $x \neq \frac{1}{2}$ and $x = \frac{1}{2}$.
1) Let $x \neq \frac{1}{2}$. Then $\pi_{x,\phi}$ is irreducible and $\{\pi_{x,\phi}(b),\ b \in D_{-1}\} = M_2(\mathbb{C})$ implying that

$$\{\pi_{x,\phi}(b),\ b \in D_{-1}\}' = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},\ \lambda \in \mathbb{C} \right\} = A_0'(x,\phi).$$

2) Let $x = \frac{1}{2}$.

2a) If $\phi = 0$ then $\pi_{\frac{1}{2},0}(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The boundary conditions for $f \in A_0$ imply that $f(\frac{1}{2},0) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{C}$. Therefore,

$$\left\{ \pi_{\frac{1}{2},0}(D_{-1}) \right\}' = A_0'(1/2,0).$$

2b) Let $\phi \in (0,2\pi),\ V = V(\phi)$ defined by (2.4). Then

$$C \in \left\{ \pi_{\frac{1}{2},\phi}(D_{-1}) \right\}' \Leftrightarrow V^*CV \in \left\{ V^*\pi_{\frac{1}{2},\phi}(D_{-1})V \right\}' \Leftrightarrow$$
$$\Leftrightarrow V^*CV \in \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},\ \lambda, \mu \in \mathbb{C} \right\}.$$

Analogously,

$$C \in A_0'((1/2,\phi)) \Leftrightarrow V^*CV \in \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},\ \lambda, \mu \in \mathbb{C} \right\}.$$

So we have $\widehat{A_0} = A_0$, and therefore D_{-1} is isomorphic to A_0. The quotient map by the equivalence (2.6) induces the isomorphism $A_0 \simeq A_1$.

The proof is completed. □

Remark 2.3. From Theorem 2.2 one can get also a description of $P(D_{-1})$, the dual space of D_{-1} (compare with [**NN**, Theorem 1.1.])

Indeed, the isomorphism $D_{-1} \simeq A_0$ shows that the $P_2(D_{-1})$, i.e the space of all pairwise non-equivalent 2-dimensional irreducible representations of D_{-1}, is homeomorphic to the open disk $D^2 \backslash \partial D^2$.

The injective map from the space of pairs $(\rho_{\frac{\phi}{2}}, \rho_{\frac{\phi}{2}+\pi})$ of 1-dimensional representations into the space of all 2 dimensional representations, $T_2(D_{-1})$, induces a covering over the circle $S^1 = \partial D^2$. Our topology on the dual space $P(D_{-1})$ can be described as follows. Let $\mathcal{I}_2$ denote this covering (over S^1 with the structure group and fiber $\mathbb{Z}_2$). Then $P(D_{-1}) = \mathcal{I}_2 \sqcup (D^2 \backslash \partial D^2)$, and a neighborhood of every $x \in \mathcal{I}_2$ is the same as for $p(x) \in S^1 = \partial D^2$ (it implies that only points from the same fiber are non-separable, i.e. for any such point there is no neighborhood which does not contain the other point).

So, to define the topology on the $P(D_{-1})$ we have only to determine the class of isomorphism of the covering $\mathcal{I}_2$. From the formulas for representations one can see, that the total space of covering is homeomorphic to S^1, so it is the unique non-trivial one (the trivial coincides with $S^1 \sqcup S^1$).

2.2. Let us consider the case of general d. Using an algorithm described in [**P1**] one gets the following

Proposition 2.4. *Any irreducible representation of WTCAR is unitarily equivalent to one of the following*

- 2^d*-dimensional representations* $\pi_{x,\phi}$, $x \in [0,1/2),\ \phi \in [0,2\pi)$,

$$
\begin{aligned}
\pi_{x,\phi}(a_i) &= \bigotimes_{1\le j<i} \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \bigotimes_{i<j\le d} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ i = 1,\ldots,d-1, \\
\pi_{x,\phi}(a_d) &= \bigotimes_{1\le j<d} \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi}\sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}.
\end{aligned} \tag{2.7}
$$

- 2^{d-1}*-dimensional representations* ρ_ϕ, $\phi \in [0, 2\pi)$,

$$
\begin{aligned}
\rho_\phi(a_i) &= \bigotimes_{1\le j<i} \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \bigotimes_{i<j<d} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ i = 1,\ldots,d-1, \\
\rho_\phi(a_d) &= \frac{e^{i\phi}}{\sqrt{2}} \bigotimes_{1\le j<d} \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix}.
\end{aligned} \tag{2.8}
$$

Moreover for different pairs (x, ϕ), $x \in (0, 1/2)$, *the corresponding representations* $\pi_{x,\phi}$ *are non-equivalent;* $\pi_{0,\phi_1} \overset{u}{\sim} \pi_{0,\phi_2}$ *for any* $\phi_1, \phi_2 \in [0, 2\pi)$*; for different* ϕ *the representations* ρ_ϕ *are non-equivalent.*

Remark 2.5. Letting $x = \frac{1}{2}$ in (2.7) we get also a representation, $\pi_{\frac{1}{2},\phi}$, of WTCAR but a reducible one:

$$\pi_{\frac{1}{2},\phi} \overset{u}{\sim} \rho_{\frac{\phi}{2}} \oplus \rho_{\frac{\phi}{2}+\pi}.$$

Theorem 2.6. *The enveloping C^*-algebra $C^*(B_\mu^{(d)})$ for WTCAR is isomorphic to* $M_{2^{d-1}}(\mathbb{C}) \otimes D_{-1}$.

Proof. As for the case $d = 1$, the enveloping C^*-algebra coincides with the C^*-algebra generated by matrix-functions on $I_{\frac{1}{2}} \times S^1$ given by (2.7). From (2.7) it also follows that

$$C^*(\pi_{x,\phi}(a_i),\ i = 1\ldots d-1) = \bigotimes_{i=1}^{d-1} M_2(\mathbb{C}) \otimes \mathbf{1},$$

and

$$\bigotimes_{i=1}^{d-1} \mathbf{1} \otimes \begin{pmatrix} 0 & e^{i\phi}\sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix} \in C^*(\pi_{x,\phi}(a_i),\ i = 1\ldots d) = C^*(\mathcal{B}_\mu^{(d)})$$

giving $C^*(\mathcal{B}_\mu^{(d)}) = M_{2^{d-1}}(\mathbb{C}) \otimes D_{-1}$. □

It follows from Theorem 2 that the dual space for $C^*(\mathcal{B}_\mu^{(d)})$ is the same as for the algebra D_{-1}. We have also

Corollary 2.7. $C^*(\mathcal{B}_\mu^{(d)}) \simeq C^*(\mathcal{B}_0^{(d)})$, $0 < \mu < 1$.

3. Representations of WCAR with two degrees of freedom

In this section we study representations of WCAR with $d = 2$ and describe the corresponding families of C^*-algebras. Let $\mathcal{E}(-1)$ denote the enveloping C^*-algebra of WCAR for $d = 2$:

$$\mathcal{E}(-1) = C^*\langle a_i^* a_i + a_i a_i^* = 1, i = 1, 2,\ a_1^* a_2 = -a_2 a_1^*\rangle. \tag{3.1}$$

Note, that WCAR is $*$-bounded: $\|\pi(a_i)\| \leq 1$ for each representation π and $i = 1, 2$.

Let us first describe irreducible representations of $\mathcal{E}(-1)$. In what follows if π is a representation of WCAR, we write simply a_i instead of $\pi(a_i)$, when no confusion can arise.

Let $A = a_2a_1 + a_1a_2$, then $A^*A = AA^*$ and $a_i^*A = Aa_i^*,\ i = 1, 2$, which allows us to apply Fuglede's theorem (see, e.g., [**R**]) and get $Aa_i = a_iA$, $i = 1, 2$. So in irreducible representation we must have

$$a_1a_2 + a_2a_1 = y \tag{3.2}$$

for some $y \in \mathbb{C}$. Moreover we will show below that $|y| \leq 1$.

In the sequel we denote by $\mathcal{E}(-1, y)$ the quotient

$$\mathcal{E}(-1)/\langle a_1a_2 + a_2a_1 - y\rangle.$$

3.1. Representations of $\mathcal{E}(-1, y)$, $y \neq 0$. It is easy to check that a_1^2 is normal and $a_i^*a_1^2 = a_1^2a_i^*,\ i = 1, 2$. Then by Fuglede's theorem, $a_1^2a_i = a_ia_1^2$, and Schur's lemma implies that $a_1^2 = \rho_1e^{i\phi_1}$.

Note that in irreducible representation of $\mathcal{E}(-1)$ we have either $a_1^2 = (a_1^*)^2 = 0$ or $\ker a_1 = \ker a_1^* = \{0\}$. Indeed, $\ker a_1^2 \neq \{0\} \Leftrightarrow \ker a_1 \neq \{0\}$, and $\ker a_1^2$ is invariant with respect to $a_i, a_i^*,\ i = 1, 2$.

Let $a_1 := u_1c_1$ be the polar decomposition of the operator a_1. Then relations (3.1) imply that

$$c_1^2u_1 = u_1(1 - c_1^2) \text{ and } c_1u_1 = u_1\sqrt{(1 - c_1^2)}.$$

Hence

$$a_1^2 = u_1c_1u_1c_1 = u_1^2c_1\sqrt{(1 - c_1^2)} = \rho e^{i\phi},$$

with $\rho \neq 0$ if $a_1^2 \neq 0$.

a) If $a_1^2 = 0$ then $a_1^*a_1 + a_1a_1^* = 1$ implies that

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}.$$

b) If $a_1^2 \neq 0$ then by uniqueness of the polar decomposition

$$u_1^2 = e^{i\phi_1} \cdot \mathbf{1},\ c_1^2(1 - c_1^2) = \rho_1^2 \cdot \mathbf{1}$$

implying that $\sigma(c_1^2) = \{x_1, 1 - x_1\}$, where $x_1(1 - x_1) = \rho_1^2$, $0 < x_1 \leq 1/2$.

Further, if $\sigma(c_1^2) = \left\{\frac{1}{2}\right\}$ then $a_1 = \frac{1}{\sqrt{2}}u_1$, where u_1 is unitary, and by Fuglede's theorem $a_2a_1 + a_1a_2 = 0$. So when $y \neq 0$ we should consider only the case $0 < x_1 < \frac{1}{2}$. Then, up to unitary equivalence,

$$a_1 = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \cdot \mathbf{1} \\ \sqrt{1 - x_1} \cdot \mathbf{1} & 0 \end{pmatrix},\ \phi_1 \in [0, 2\pi). \tag{3.3}$$

We start with the analysis of case **b)** and suppose that a_1 is given by (3.3)

From $a_1^*a_2 = -a_2a_1^*$ we get

$$a_2 = \begin{pmatrix} b_1 & b_2\sqrt{1 - x_1} \\ -e^{-i\phi_1}b_2\sqrt{x_1} & -b_1 \end{pmatrix},$$

then the relation $a_1a_2 + a_2a_1 = y$ implies that $b_2(1-2x_1) = y$. Since $0 < x_1 < \frac{1}{2}$,

$$b_2 = \frac{y}{1-2x_1},\ a_2 = \begin{pmatrix} b_1 & \frac{y}{1-2x_1}\sqrt{1-x_1} \\ -e^{-i\phi_1}\frac{y}{1-2x_1}\sqrt{x_1} & -b_1 \end{pmatrix},$$

and from $a_2^*a_2 + a_2a_2^* = 1$ we get

$$b_1^*b_1 + b_1b_1^* = 1 - \frac{|y|^2}{(1-2x_1)^2}. \tag{3.4}$$

Clearly the family $\{a_i, a_i^*,\ i=1,2\}$ is irreducible iff so is $\{b_1, b_1^*\}$.

Representations of (3.4) exist iff $|y| \le 1 - 2x_1$. Furthermore $x_1 \in (0, 1/2)$ implies $|y| \le 1$ and $x_1 \le \frac{1-|y|}{2}$. Evidently, $0 < x_1 \le \frac{1-|y|}{2}$, $|y| \le 1$ yields $|y| < 1$.

b1) If $x_1 = \frac{1-|y|}{2}$, then $b_1 = 0$.

b2) If $0 < x_1 < \frac{1-|y|}{2}$ put $b_1 = \sqrt{1 - \frac{|y|^2}{(1-2x_1)^2}}\cdot\widetilde{b}_1$, then $\widetilde{b}_1^*\widetilde{b}_1 + \widetilde{b}_1\widetilde{b}_1^* = 1$ and the family $\{a_i,\ a_i^*,\ i=1,2\}$ is irreducible iff so is $\left\{\widetilde{b}_1,\ \widetilde{b}_1^*\right\}$. Two families $\left\{a_i^{(j)},\ a_i^{(j)*},\ i=1,2\right\}$, $j=1,2$, are unitarily equivalent iff the corresponding families $\left\{\widetilde{b}_1^{(j)},\ \widetilde{b}_1^{(j)*}\right\}$, $j=1,2$, are unitarily equivalent.

Then in the irreducible case one has (see Section 2)

b2.1)

$$b_1 = \sqrt{1 - \frac{|y|^2}{(1-2x_1)^2}}\begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix}, \tag{3.5}$$

$$0 \le x_2 < \frac{1}{2},\ \phi_2 \in [0, 2\pi).$$

b2.2)

$$b_1 = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{|y|^2}{(1-2x_1)^2}}e^{i\phi_2},\ \phi_2 \in [0, 2\pi). \tag{3.6}$$

We now turn to case **a)**: $x_1 = 0$. Repeating the arguments given in **b)** with $x_1 = 0$ we obtain that

$$a_1 = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} b_1 & y \\ 0 & -b_1 \end{pmatrix} \text{ and } b_1^*b_1 + b_1b_1^* = 1 - |y|^2.$$

If $|y| = 1$, then $b_1 = 0$. When $|y| < 1$ to get b_1 one has to put $x_1 = 0$ in (3.5), (3.6).

Let

$$D_y = \left\{(x_1, x_2) |\ 0 \le x_1 \le \frac{1-|y|}{2},\ 0 \le x_2 \le \frac{1}{2}\right\}$$

Define an equivalence on $D_y \times \mathbf{T}^2$:

$$(x_1, x_2, e^{i\phi_1}, e^{i\phi_2}) \sim (y_1, y_2, e^{\psi_1}, e^{i\psi_2})$$

iff either $x_1 = y_1 = 0$, $x_2 = y_2$, $\phi_2 = \psi_2$ or $x_2 = y_2 = 0$, $x_1 = y_1$, $\phi_1 = \psi_1$ or $x_1 = y_1 = \frac{1-|y|}{2}$, $\phi_1 = \psi_1$.

Summarizing the above discussion we have

THEOREM 3.1. *The C^*-algebra $\mathcal{E}(-1, y)$ is non-zero iff $|y| \le 1$. If $y \ne 0$, then any irreducible representation of $\mathcal{E}(-1, y)$ is unitarily equivalent to one of the following:*

a) *If* $|y|=1$, $a_1=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $a_2=\begin{pmatrix}0&y\\0&0\end{pmatrix}$.

b) $|y|<1$.

b1) *If* $x_1=\frac{1-|y|}{2}$, *then*

$$a_1=\begin{pmatrix}0&e^{i\phi_1}\sqrt{\frac{1-|y|}{2}}\\\sqrt{\frac{1+|y|}{2}}&0\end{pmatrix},\ a_2=\begin{pmatrix}0&\frac{y}{|y|}\sqrt{\frac{1+|y|}{2}}\\-\frac{e^{-i\phi_1}y}{|y|}\sqrt{\frac{1-|y|}{2}}&0\end{pmatrix}.$$

b2) *If* $0\le x_1<\frac{1-|y|}{2}$, *then*

b2.1) *if* $0\le x_2<\frac{1}{2}$,

$$a_1=\begin{pmatrix}0&e^{i\phi_1}\sqrt{x_1}\\\sqrt{1-x_1}&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \phi_1\in[0,2\pi),\ \phi_2\in[0,2\pi)$$

$$a_2=\sqrt{1-\frac{|y|^2}{(1-2x_1)^2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}0&e^{i\phi_2}\sqrt{x_2}\\\sqrt{1-x_2}&0\end{pmatrix}+ \tag{3.7}$$

$$+\frac{y}{1-2x_1}\begin{pmatrix}0&\sqrt{1-x_1}\\-e^{-i\phi_1}\sqrt{x_1}&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix};$$

b2.2) *if* $x_2=\frac{1}{2}$,

$$a_1=\begin{pmatrix}0&e^{i\phi_1}\sqrt{x_1}\\\sqrt{1-x_1}&0\end{pmatrix},$$

$$a_2=\begin{pmatrix}\frac{e^{i\phi_2}}{\sqrt{2}}\sqrt{1-\frac{|y|^2}{(1-2x_1)^2}}&\frac{y\sqrt{1-x_1}}{1-2x_1}\\-e^{-i\phi_1}\frac{y\sqrt{x_1}}{1-2x_1}&-\frac{e^{i\phi_2}}{\sqrt{2}}\sqrt{1-\frac{|y|^2}{(1-2x_1)^2}}\end{pmatrix},\ \phi_1,\phi_2\in[0,2\pi).$$

When $y\ne0,\ |y|<1$, *is fixed, the representations corresponding to non-equivalent tuples* $(x_1,x_2,e^{i\phi_1},e^{i\phi_2})\in D_y\times\mathbf{T}^2$ *are non-equivalent.*

Using this classification we can describe $\mathcal{E}(-1,y)$ for $0<|y|\le1$. The case $y=0$ will be studied separately below.

THEOREM 3.2. ***1)*** *If* $0<|y|<1$ *then* $\mathcal{E}(-1,y)$ *can be realized as follows*

$$\begin{gathered}\mathcal{E}(-1,y)=\big\{f\in C(D_y\times\mathbf{T}^2\to M_4(\mathbb{C}))\mid\\ f(0,x_2,e^{i\phi_1},e^{i\phi_2})=f(0,x_2,1,e^{i\phi_2}),\\ f(x_1,0,e^{i\phi_1},e^{i\phi_2})=f(x_1,0,e^{i\phi_1},1),\\ f\big((1-|y|)/2,x_2,e^{i\phi_1},e^{i\phi_2}\big)=f\big((1-|y|)/2,0,e^{i\phi_1},1\big)\in M_2(\mathbb{C})\otimes\mathbf{1},\\ \nu_2^*(\phi_2)f(x_1,1/2,e^{i\phi_1},e^{i\phi_2})\nu_2(\phi_2)\in M_2(\mathbb{C})\oplus M_2(\mathbb{C}),\ \forall\phi_2\in[0,2\pi]\big\},\end{gathered}$$

where

$$\begin{gathered}\nu_2:[0,2\pi]\to U(4),\ \phi\mapsto T(V(\phi)\otimes\mathbf{1}_2),\\ T:\mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}^2\otimes\mathbb{C}^2,\ x\otimes y\mapsto y\otimes x,\quad V(\phi)\ \textit{is given by (2.4).}\end{gathered}$$

2) If $|y| = 1$ then $\mathcal{E}(-1, y)$ is isomorphic to $M_2(\mathbb{C})$.

Proof. First, note that any irreducible representation of $\mathcal{E}(-1, y)$ is either defined by formulas (3.7) or can be obtained by decomposition in irreducible components of such representations with $x_1 = \frac{1-|y|}{2}$ or $x_2 = \frac{1}{2}$. Hence if a_i are given by (3.7), $C^*\big(a_i(x_1, x_2, e^{i\phi_1}, e^{i\phi_2}),\ i = 1, 2\big)$ is isomorphic to $\mathcal{E}(-1, y)$. Further, it is easy to check that

$$\nu_2^*(\phi_2) a_1(x_1, 1/2, e^{i\phi_1}, e^{i\phi_2}) \nu_2(\phi_2) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1-x_1} & 0 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

$$\nu_2^*(\phi_2) a_2(x_1, 1/2, e^{i\phi_1}, e^{i\phi_2}) \nu_2(\phi_2) =$$

$$= \frac{1}{\sqrt{2}} \sqrt{1 - \frac{|y|^2}{(1-2x_1)^2}} \begin{pmatrix} e^{i\frac{\phi_2}{2}} & 0 \\ 0 & -e^{i\frac{\phi_2}{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} +$$

$$+ \frac{y}{1-2x_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \sqrt{1-x_1} \\ e^{-i\phi_1} & 0 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

$$a_1\big((1-\mid y \mid)/2, x_2, e^{i\phi_1}, e^{i\phi_2}\big) = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{\frac{1-|y|}{2}} \\ \sqrt{\frac{1+|y|}{2}} & 0 \end{pmatrix} \otimes \mathbf{1}_2,$$

$$a_2\big((1-\mid y \mid)/2, x_2, e^{i\phi_1}, e^{i\phi_2}\big) = \begin{pmatrix} 0 & \frac{y}{|y|}\sqrt{\frac{1+|y|}{2}} \\ -\frac{e^{-i\phi_1} y}{|y|}\sqrt{\frac{1-|y|}{2}} & 0 \end{pmatrix} \otimes \mathbf{1}_2.$$

and if $\pi_j,\ j = 1, 2$, are representations (possibly reducible) defined by (3.7) with non-equivalent tuples of parameters then $C(\pi_1, \pi_2) = \{0\}$. To complete the proof use the same arguments as in the proof of Theorem 1. $\square$

3.2. Representations and enveloping C^*-algebra of $\mathcal{E}(-1, 0)$. Let us consider the case $y = 0$. Analysis similar to that in Subsection 3.1 gives a description of the unitary equivalence classes of irreducible representations of $\mathcal{E}(-1, 0)$. Note that here one must also consider the case $\sigma(c_1^2) = \left\{\frac{1}{2}\right\}$. Namely, we have that the generators a_1, a_2 in irreducible representations of $\mathcal{E}(-1, 0)$ have one of the forms presented below.

1)

$$a_1 = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1-x_1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(3.8)

$$a_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix},$$

$$0 \le x_1, x_2 < 1/2,\ 0 \le \phi_1, \phi_2 < 2\pi;$$

2)

2a)

$$a_1 = \frac{e^{i\phi_1}}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix}, \tag{3.9}$$

$$0 \le x_2 < 1/2,\ 0 \le \phi_1, \phi_2 < 2\pi,$$

2b)

$$a_1 = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1-x_1} & 0 \end{pmatrix}, a_2 = \frac{e^{i\phi_2}}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$0 \le x_1 < 1/2,\ 0 \le \phi_1, \phi_2 < 2\pi;$$

3)

$$a_1 = \frac{e^{i\phi_1}}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{i\phi_2} \\ 1 & 0 \end{pmatrix}, \tag{3.10}$$

$$0 \le \phi_1 < \pi,\ 0 \le \phi_2 < 2\pi.$$

As above we introduce an equivalence on $I_{\frac{1}{2}} \times I_{\frac{1}{2}} \times \mathbf{T}^2$:

$$(x_1, x_2, e^{i\phi_1}, e^{i\phi_2}) \sim (y_1, y_2, e^{i\psi_2}, e^{i\psi_2}),$$

iff $x_1 = y_1 = 0,\ x_2 = y_2,\ \phi_2 = \psi_2$ or $x_2 = y_2 = 0,\ x_1 = y_1,\ \phi_1 = \psi_1$. The representations given by **1), 2), 3)** are equivalent iff they correspond to equivalent quadruples of parameters.

We next present a realization of $\mathcal{E}(-1,0)$ analogous to that given in Section 3.1.

THEOREM 3.3. *The C^*-algebra $\mathcal{E}(-1,0)$ is isomorphic to the following algebra of continuous matrix-functions*

$$\begin{gathered} \big\{f \in C(I_{\frac{1}{2}} \times I_{\frac{1}{2}} \times S^1 \times S^1 \to M_4(\mathbb{C})) \mid \\ f(0, x_2, e^{i\phi_1}, e^{i\phi_2}) = f(0, x_2, 1, e^{i\phi_2}), \\ f(x_1, 0, e^{i\phi_1}, e^{i\phi_2}) = f(x_1, 0, e^{i\phi_1}, 1), \\ \nu_1^*(\phi_1, \phi_2) f(1/2, x_2, e^{i\phi_1}, e^{i\phi_2}) \nu_1(\phi_1, \phi_2) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),\ \forall \phi_1, \phi_2 \in [0, 2\pi], \\ \nu_2^*(\phi_2) f(x_1, 1/2, e^{i\phi_1}, e^{i\phi_2}) \nu_2(\phi_2) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), \forall \phi_2 \in [0, 2\pi]\big\}, \end{gathered}$$

where

$$\nu_1 : [0, 2\pi] \times [0, 2\pi] \to U(4),\ (\phi_1, \phi_2) \mapsto W(\phi_2)(V(\phi_1) \otimes \mathbf{1}_2),$$
$$\nu_2 : [0, 2\pi) \to U(4),\ \phi_2 \mapsto T(V(\phi_2) \otimes \mathbf{1}_2),$$
$$T : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2,\ x \otimes y \mapsto y \otimes x,$$
$$W(\phi) = (\mathbf{1}_2 \otimes V(\phi))S(\mathbf{1}_2 \otimes V^*(\phi)),\ V(\phi) \textit{ is given by (2.4)},$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. The proof is similar to ones of Theorems 2.2 and 3.2. We note only that

$$\nu_1^*(\phi_1, \phi_2)a_1(1/2, x_2, e^{i\phi_1}, e^{i\phi_2})\nu_1(\phi_1, \phi_2) =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\phi_1}{2}} & 0 \\ 0 & e^{-i\frac{\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

$$\nu_1^*(\phi_1, \phi_2)a_2(1/2, x_2, e^{i\phi_1}, e^{i\phi_2})\nu_1(\phi_1, \phi_2) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1 - x_2} & 0 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

$$\nu_2^*(\phi_2)a_1(x_1, 1/2, e^{i\phi_1}, e^{i\phi_2})\nu_2(\phi_2) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1 - x_1} & 0 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

$$\nu_2^*(\phi_2)a_2(x_1, 1/2, e^{i\phi_1}, e^{i\phi_2})\nu_2(\phi_2) =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\phi_2}{2}} & 0 \\ 0 & e^{-i\frac{\phi_2}{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

and $C(\pi_1, \pi_2) = \{0\}$ if π_1, π_2 are determined by (3.8) with non-equivalent tuples of parameters. ☐

Next we study when the C^*-algebras $\mathcal{E}(-1, y)$ are isomorphic.

Proposition 3.4. *For any* y_1, y_2, *such that either* $0 < |y_i| < 1$, $i = 1, 2$, *or* $|y_i| = 1$, $i = 1, 2$, $\mathcal{E}(-1, y_1)$ *is isomorphic to* $\mathcal{E}(-1, y_2)$. $\mathcal{E}(-1, 0)$ *is not isomorphic to any other* $\mathcal{E}(-1, y)$.

Proof. The isomorphism of $\mathcal{E}(-1, y_1)$ and $\mathcal{E}(-1, y_2)$ with $0 < |y_i| < 1$, $i = 1, 2$, is induced by the homeomorphism $\vartheta\colon D_{y_1} \times \mathbf{T}^2 \to D_{y_2} \times \mathbf{T}^2$ given by the rule

$$(x_1, x_2, e^{i\phi_1}, e^{i\phi_2}) \mapsto (\frac{1 - |y_2|}{1 - |y_1|}x_1, x_2, e^{i\phi_1}, e^{i\phi_2})$$

To see that $\mathcal{E}(-1,0)$ is not isomorphic to $\mathcal{E}(-1,y)$ when $0<|y|<1$ consider the family $\rho=\{\rho_{\lambda_1,\lambda_2},\ |\lambda_i|=1,\ i=1,2\}$ of automorphisms of both algebras defined by

$$\rho_{\lambda_1,\lambda_2}(a_i)=\lambda_i a_i,\ i=1,2.$$

Denote by $\mathcal{F}_0\subset\mathcal{E}(-1,0)$ and $\mathcal{F}_y\subset\mathcal{E}(-1,y)$ the C^*-subalgebras of the elements fixed by the family ρ.

Then

$$\mathcal{F}_0=C^*(a_i^n(a_i^*)^n,\ (a_i^*)^n a_i^n,\ i=1,2,\ n\in\mathbb{Z}_+).$$

In fact, given $b\in\mathcal{F}_0,\ \varepsilon>0$, there exists a polynomial p in $a_i,\ a_i^*$ such that $\|b-p\|<\varepsilon$. We can decompose p into sum of two polynomials p_1, p_2 such that p_1 is a sum of those homogeneous terms of p where each a_i appears so many times as a_i^* and $p_2=p-p_1$. Using the relations in the algebra we have

$$p_1\in C^*(a_i^n(a_i^*)^n,\ (a_i^*)^n a_i^n,\ i=1,2,\ n\in\mathbb{Z}_+).$$

As

$$\int_0^{2\pi}\int_0^{2\pi}\rho_{\lambda_1,\lambda_2}(p_2)d\lambda_1 d\lambda_2=0,$$

$$\begin{aligned}\|b-p_1\|&=\frac{1}{(2\pi)^2}\|\int_0^{2\pi}\int_0^{2\pi}\rho_{\lambda_1,\lambda_2}(b-p)d\lambda_1 d\lambda_2\|\le\\&\le\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\|\rho_{\lambda_1,\lambda_2}(b-p)\|d\lambda_1 d\lambda_2=\\&=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\|b-p\|d\lambda_1 d\lambda_2=\|b-p\|<\varepsilon,\end{aligned}$$

and hence $b\in C^*(a_i^n(a_i^*)^n,\ (a_i^*)^n a_i^n,\ i=1,2,\ n\in\mathbb{Z}_+)$. These arguments are standard in operator algebras theory and we give them just for the completeness.

It follows from the relations between generators of $\mathcal{E}(-1,0)$ and the description of $\mathcal{F}_0$ that $\mathcal{F}_0$ is commutative. However, $\mathcal{F}_y$ is non-commutative, since, for example, in $\mathcal{E}(-1,y)$, $a_1a_1^*a_2a_2^*\neq a_2a_2^*a_1a_1^*$. Hence $\mathcal{F}_0\not\simeq\mathcal{F}_y$ and therefore $\mathcal{E}(-1,0)\not\simeq\mathcal{E}(-1,y)$ when $0<|y|<1$. To complete the proof recall that $\mathcal{E}(-1,y)$ is isomorphic to $M_2(\mathbb{C})$ if $|y|=1$ while $\mathcal{E}(-1,0)$ is not. □

Finally, let us consider the family of C^*-algebras, $\mathcal{E}(\varepsilon)$, defined as enveloping for WCAR with

$$a_2a_1+a_1a_2=y,\ \varepsilon\le|\,y\,|\le 1.$$

It will be convenient for us to consider the polar decomposition of y, $y=re^{i\phi}$. Using the results of Theorem 3.2 it is easy to get the description of $\mathcal{E}(\varepsilon)$. Let

$$D_\varepsilon=\{(r,x_1,x_2)\mid\ \varepsilon\le r\le 1,\ 0\le x_1\le\frac{1-r}{2},\ 0\le x_2\le\frac{1}{2}\}.$$

THEOREM 3.5. *For any $0 < \varepsilon < 1$ the C^*-algebra $\mathcal{E}(\varepsilon)$ can be realized as follows:*

$$\begin{gathered}\mathcal{E}(\varepsilon) = \big\{ f \in C(D_\varepsilon \times \mathbf{T}^3 \to M_4(\mathbb{C})) \mid \\ f(r,0,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2}) = f(r,0,x_2,e^{i\phi},1,e^{i\phi_2}), \\ f(r,x_1,0,e^{i\phi},e^{i\phi_1},e^{i\phi_2}) = f(r,x_1,0,e^{i\phi},e^{i\phi_1},1), \\ f\big(r,(1-r)/2,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2}\big) = f\big(r,(1-r)/2,0,e^{i\phi},e^{i\phi_1},1\big) \in M_2(\mathbb{C}) \otimes \mathbf{1}, \\ f\big(1,0,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2}\big) = f\big(1,0,0,e^{i\phi},1,1\big), \\ \nu_2^*(\phi_2) f(r,x_1,1/2,e^{i\phi},e^{i\phi_1},e^{i\phi_2}) \nu_2(\phi_2) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), \forall\ \phi_2 \in [0,2\pi] \big\}.\end{gathered}$$

PROOF. The proof is evident. □

PROPOSITION 3.6. *For any ε_1, ε_2, $0 < \varepsilon_i < 1$, $i = 1,2$, $\mathcal{E}(\varepsilon_1)$ is isomorphic to $\mathcal{E}(\varepsilon_2)$.*

PROOF. The required isomorphism is induced by the following homeomorphism $\vartheta\colon D_{\varepsilon_1} \times \mathbf{T}^3 \to D_{\varepsilon_2} \times \mathbf{T}^3$:

$$(r,x_1,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2}) \mapsto (\varepsilon_2 + \frac{1-\varepsilon_2}{1-\varepsilon_1}(r-\varepsilon_1), \frac{1-\varepsilon_2}{1-\varepsilon_1} x_1, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}).$$

□

4. Description of $\mathcal{E}(-1)$.

Our goal in this section to describe the "global" C^*-algebra. We put $y := re^{i\phi}$ and $r_1 := \frac{r}{1-2x_1}$. When $0 \le r \le 1$ and $0 \le x_1 \le \frac{1-r}{2}$ one has $0 \le r_1 \le 1$. So, representation given by (3.7) has the form

$$a_1 = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1-x_1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \phi_1 \in [0,2\pi), \ \phi_2 \in [0,2\pi)$$

$$a_2 = \sqrt{1-r_1^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix} + \tag{4.1}$$

$$+ r_1 e^{i\phi} \begin{pmatrix} 0 & \sqrt{1-x_1} \\ -e^{-i\phi_1}\sqrt{x_1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

Consider

$$D = \{(r_1,x_1,x_2) \ : \ 0 \le x_i \le \frac{1}{2}, \ i = 1,2, \ 0 \le r_1 \le 1\} \times \mathbf{T}^3$$

In this section we show that $\mathcal{E}(-1)$ is isomorphic to the C^*-algebra of continuous matrix-functions on D generated by a_1, a_2 given by (4.1). To do so we need the following auxiliary lemma.

LEMMA 4.1. *Let π be a representation defined by (4.1) corresponding to a tuple $(r_1, 1/2, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2})$, where $0 < r_1 \le 1$. Then π is equivalent to representation $\widetilde{\pi}$ defined by (4.1) with parameters $(0, 1/2, t, 1, e^{i\phi_1}, e^{i\psi})$, where*

$$\sqrt{t(1-t)} = | z |, \ \psi = arg\ z,$$

$$z = e^{i\phi_2}(1-r_1^2)\sqrt{x_2(1-x_2)} - \frac{r_1^2 e^{i(2\phi-\phi_1)}}{2}$$

Proof. First we note that the representation $\widetilde{\pi}$ corresponding to $(0,1/2,t,1,e^{i\phi_1},e^{i\psi})$ coincides with a representation of $\mathcal{E}(0,-1)$ given by (3.8) with $x_1=1/2$, ϕ_1 and $\phi_2=\psi$ and it is equivalent to the direct sum of two-dimensional irreducible representations given by formula (3.9), Section 3.2.:

$$\widetilde{\pi}\sim\pi_1\oplus\pi_2,$$

where π_1 corresponds to parameters $x_2=t$, $\phi_1/2$ and $\phi_2=\psi$ and π_2 corresponds to $(t,\frac{\phi_1}{2}+\pi,\psi)$. It is easy to see that $\pi_1\not\sim\pi_2$ if $t\neq\frac{1}{2}$ and $\pi_1\sim\pi_2$ otherwise.

Thus, to prove lemma we have to show that for tuple $(r_1,1/2,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2})$, $0<r_1\le 1$ the corresponding representation π is equivalent to $\pi_1\oplus\pi_2$.

Since $a_1,\ a_2$ defined by (4.1) with $x_1=\frac{1}{2}$ give a reducible representation of WCAR and satisfy $a_1a_2+a_2a_1=0$, according to results of Sec. 3.2, π is equivalent to the direct sum of two-dimensional irreducible representations of $\mathcal{E}(-1,0)$, $\pi\sim\widehat{\pi}_1\oplus\widehat{\pi}_2$. Further

$$a_2^2=e^{i\phi_2}(1-r_1^2)\sqrt{x_2(1-x_2)}-\frac{r_1^2e^{i(2\phi-\phi_1)}}{2}:=z$$

implies that $\sigma(a_2^*a_2)=\{t,1-t\}$, where $\sqrt{t(1-t)}=\mid z\mid$ and $u_2^2=e^{i\psi}$, $\psi=arg\ z$, where $a_2=u_2(a_2^*a_2)^{\frac{1}{2}}$ is a polar decomposition.

If $\mid z\mid<\frac{1}{2}$ then $0\le t<\frac{1}{2}$, and taking into account $a_1^2=\frac{e^{i\phi_1}}{2}$ we conclude that in this case $\widehat{\pi}_1$, $\widehat{\pi}_2$ are determined by formulas (3.9), Sec. 3.2, with $x_2=t$, $\phi_2=\psi$. To describe the decomposition of π completely it is remained to verify whether or not $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are equivalent. To do so we compute the dimension of commutant of π. It is a routine to verify that if $\mid z\mid<\frac{1}{2}$ then

$$\dim\{\pi(a_i),\ i=1,2\}'=2.$$

Hence $\widehat{\pi}_1\not\sim\widehat{\pi}_2$ and we can suppose that $\widehat{\pi}_1$, $\widehat{\pi}_2$ correspond to (3.9) with tuples $(t,\frac{\phi_1}{2},\psi)$ and $(t,\frac{\phi_1}{2}+\pi,\psi)$ respectively. I.e., if $\mid z\mid<\frac{1}{2}$, then $\widehat{\pi}_i\sim\pi_i$, $i=1,2$.

If $\mid z\mid=\frac{1}{2}$ then $t=\frac{1}{2}$, $\psi=2\phi-\phi_1+\pi\mod 2\pi$, and

$$a_1^2=\frac{e^{i\phi_1}}{2},\ a_2^2=\frac{e^{i(2\phi-\phi_1+\pi)}}{2}. \tag{4.2}$$

(Moreover if $0<r_1<1$ we additionally have $x_2=\frac{1}{2}$ and $\phi_2=2\phi-\phi_1+\pi\mod 2\pi$.)

Since the two-dimensional irreducible representation of $\mathcal{E}(0,-1)$ with $a_1,\ a_2$ satisfying (4.2) is unique we conclude that $\widehat{\pi}_1\sim\widehat{\pi}_2$ and defined by (3.10), Sec. 3.2, with $(\frac{\phi_1}{2},\psi)$. Hence, when $\mid z\mid=\frac{1}{2}$ we also have $\widehat{\pi}_i\sim\pi_i$, $i=1,2$ (recall, that the representations given by (3.10) corresponding to $(\frac{\phi_1}{2},\psi)$ and $(\frac{\phi_1}{2}+\pi,\psi)$ are equivalent).

So, we have shown that there exists a unitary matrix-function

$$U=U(r_1,x_2,\phi,\phi_1,\phi_2),$$

such that

$$U^*(r_1,x_2,\phi,\phi_1,\phi_2)a_i(r_1,1/2,x_2,e^{i\phi},e^{i\phi_1},e^{i\phi_2})U(r_1,x_2,\phi,\phi_1,\phi_2)= \\ =a_i(0,1/2,t,1,e^{i\phi_1},e^{i\psi}),$$

where a_i, $i=1,2$, are defined by (4.1) and t, ψ are specified above. □

Remark 4.2. The matrix-function U can be written explicitly, however we do not give it here.

Now we are ready to formulate the main results of this section.

THEOREM 4.3.

$$\mathcal{E}(-1) \simeq C^*(a_i(r_1, x_1, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}),\ i = 1, 2)$$

where a_i, $i = 1, 2$, are given by (4.1) and $(r_1, x_1, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) \in D$.

PROOF. The result follows directly from the definition of enveloping C^*-algebra and results of Theorems 3.2, 3.3 and Lemma 4.1. □

THEOREM 4.4. *The C^*-algebra $\mathcal{E}(-1)$ is isomorphic to the C^*-algebra of continuous matrix-functions $f\colon D \to M_4(\mathbb{C})$ satisfying the following boundary conditions*

$$f(r_1, 0, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) = f(r_1, 0, x_2, e^{i\phi}, 1, e^{i\phi_2})$$

$$f(r_1, x_1, 0, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) = f(r_1, x_1, 0, e^{i\phi}, e^{i\phi_1}, 1)$$

$$f(1, x_1, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) = f(1, x_1, 0, e^{i\phi}, e^{i\phi_2}, 1) \in M_2(\mathbb{C}) \otimes \mathbf{1}_2$$

$$f(0, x_1, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) = f(0, x_1, x_2, 1, e^{i\phi_1}, e^{i\phi_2})$$

$$\nu_2^*(\phi_2) f(r_1, x_1, 1/2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) \nu_2(\phi_2) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

$$\nu_1^*(\phi_1, \phi_2) f(0, 1/2, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) \nu_1(\phi_1, \phi_2) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

$$U^*(r_1, x_2, \phi, \phi_1, \phi_2) f(r_1, 1/2, x_2, e^{i\phi}, e^{i\phi_1}, e^{i\phi_2}) U(r_1, x_2, \phi, \phi_1, \phi_2) =$$
$$= f(0, 1/2, t, 1, e^{i\phi_1}, e^{i\psi}),$$

where t, ψ and $U(\cdot)$ are specified in Lemma 4.1.

PROOF. The proof is the same as for Theorems 3.2, 3.3. □

REMARK 4.5. Note that $U(0, x_2, \phi, \phi_1, \phi_2) = U(0, x_2, 1, \phi_1, \phi_2) = \mathbf{1}_4$.

Acknowledgements. We are indebted to Prof. P.E.T. Jørgensen and Prof. Yu.S. Samoĭlenko for helpful discussions on the subject.

References

[B] L.C. Biedenharn, *The quantum group $SU_q(2)$ and a q-analogue of the boson operators*, J. Phys. A , **22** (1989), L873–L878.

[BS] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Commun. Math. Phys., **137** (1991), 519–531.

[F] Fell J.M.G., *The structure of algebras of operator fields*, Acta Math., **106**, n. 3-4, (19 61), 233–280.

[F] D.I. Fivel, *Interpolation between Fermi and Bose statistics using generalized commutators*, Phys. Rev. Lett., **65** (1990), 3361–3364.

[G] O.W. Greenberg, *Particles with small violations of Fermi and Bose statistics*, Phys. Rev. D., **43** (1991), 4111–4129.

[JSW] P. E. T. Jørgensen, L. M. Schmitt and R. F. Werner, *Positive representations of general commutation relations allowing Wick ordering*, *J. Funct. Anal.* **134**, no.1 (1995), 33–99.

[JW] P. E. T. Jørgensen and R. F. Werner. *Coherent states of the q-canonical commutation relations*, arXiv: funct-an/9303002.

[M1] I. Markov, *The C^*-algebra generated by a non-commutative circle*, Applications of the methods of functional anlysis in mathematical physics (Russian), Kiev (1991), 70–78.

[M2] W. Marcinek, *On commutation relations for quons*, Rep. Math. Phys. **41** (1998), no. 2, 155–172.

[M3] A.J. Macfarlane, *On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$*, J. Phys. A., **22** (1989), 4581–4588.

[NN] A. Nica and G. Nagy, *On the quantum disk and non-commutative circle*, Algebraic Methods in Operator Theory, P.E.T. Jørgensen and R. Curto eds., Birkhauser, Boston, 1994, 276–290.

[OS] Ostrovskyĭ, V. and Samoĭlenko, Yu., *Introduction to the Theory of Representations of Finitely Presented *-Algebras. I. Representations by bounded operators*, The Gordon and Breach Publishing group, London, 1999.

[P1] D. Proskurin, *Homogeneous ideals in Wick $*$-algebras*, Proc. Amer. Math. Soc., **126** no. 11 (1998), 3371-3376.

[P2] W. Pusz , *Twisted canonical anti-commutation relations*, Reports Math. Phys., **27** (1989), 349–360.

[R] W.Rudin, Functional analysis, Mc Graw-Hill, New York, 1973.

[TT] Tomiyama J., Takesaki M.: *Application of fibre bundles to certain class of C^*-algebras*, Tohoku Math. Journ. **13** no.3 (1961), 498–522.

[V] Vasil'ev N., *C^*-algebras with finite dimensional irreducible representations*, Uspehi mat. nauk. (in Russian) **XXI.** no.1 (1966), 136–154.

Department of Cybernetics, Kyiv Taras Shevchenko University, Volodymyrska, 64, 01033 Kyiv, Ukraine

E-mail address: prosk@univ.kiev.ua.

Department of Cybernetics, Kyiv Taras Shevchenko University, Volodymyrska, 64, 01033 Kyiv, Ukraine

E-mail address: yurius@univ.kiev.ua

Chalmers Tekniska Högskola, Matematiska Vetenskaper, SE 412 96 Göteborg, Sweden

E-mail address: turowska@math.chalmers.se